7.5 Integrals of Scalar Functions over surfaces

Example: the mass of a thin sheet of metal $S = \Phi(D)$ where $D \in \mathbb{R}^2$ & $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ where the density of the metal is given by $f(x,y,z)$

Definition: The integral of a scalar function over a surface

$$\iint_S f(x,y,z)\,dS \underset{\text{notation}}{=} \iint_S f\,dS \underset{\text{def'n}}{=} \iint_D f(\Phi(u,v))\,\|T_u \times T_v\|\,du\,dv$$

$$= \iint_D f(x(u,v), y(u,v), z(u,v)) \sqrt{\left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(u,v)}\right)^2}\,du\,dv$$

Example

$f(x,y,z) = \sqrt{x^2+y^2+z^2}$, $S: (r\cos\theta, r\sin\theta, \theta)$

$D: 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 1$

(Helicoid) (see p. 386 of book)

Compute $\iint_D f\,dS$

$$\iint_D f\,dS = \iint_D f(r\cos\theta, r\sin\theta, \theta)\,\|T_r \times T_\theta\|\,dr\,d\theta$$

$$= \iint_D \sqrt{r^2+1}\ \|T_r \times T_\theta\|\,dr\,d\theta$$

but $T_r = (\cos\theta, \sin\theta, 0)$ $\Rightarrow T_r \times T_\theta = \sin\theta\,\vec{i} - \cos\theta\,\vec{j} + r\,\vec{k}$

$T_\theta = (-r\sin\theta, r\cos\theta, 1)$

$\Rightarrow \|T_r \times T_\theta\| = \sqrt{1+r^2}$

$$\Rightarrow \iint_S f\,dS = \int_0^1\int_0^{2\pi} \left(\sqrt{1+r^2}\right)\left(\sqrt{1+r^2}\right) d\theta\,dr = 2\pi\left(r + \frac{r^3}{3}\right)\Big|_0^1 =$$

$$= 8\pi/3$$

Surface integrals over graphs of functions

Suppose S is the graph of a differentiable function $z = g(x,y)$

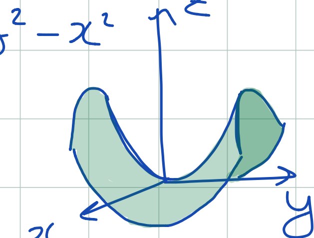
$\Rightarrow$ we can parametrize as $(u, v, g(u,v))$

$\Rightarrow \|T_u \times T_v\| = \sqrt{1 + \left(\frac{\partial g}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2}$

So $$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\,dx\,dy$$

Example: Compute $\iint_S \frac{x}{\sqrt{4x^2+4y^2+1}}\,dS$ where

S is the hyperbolic paraboloid $z = y^2 - x^2$ over the region $-1 \le y \le 1$, $-1 \le x \le 1$



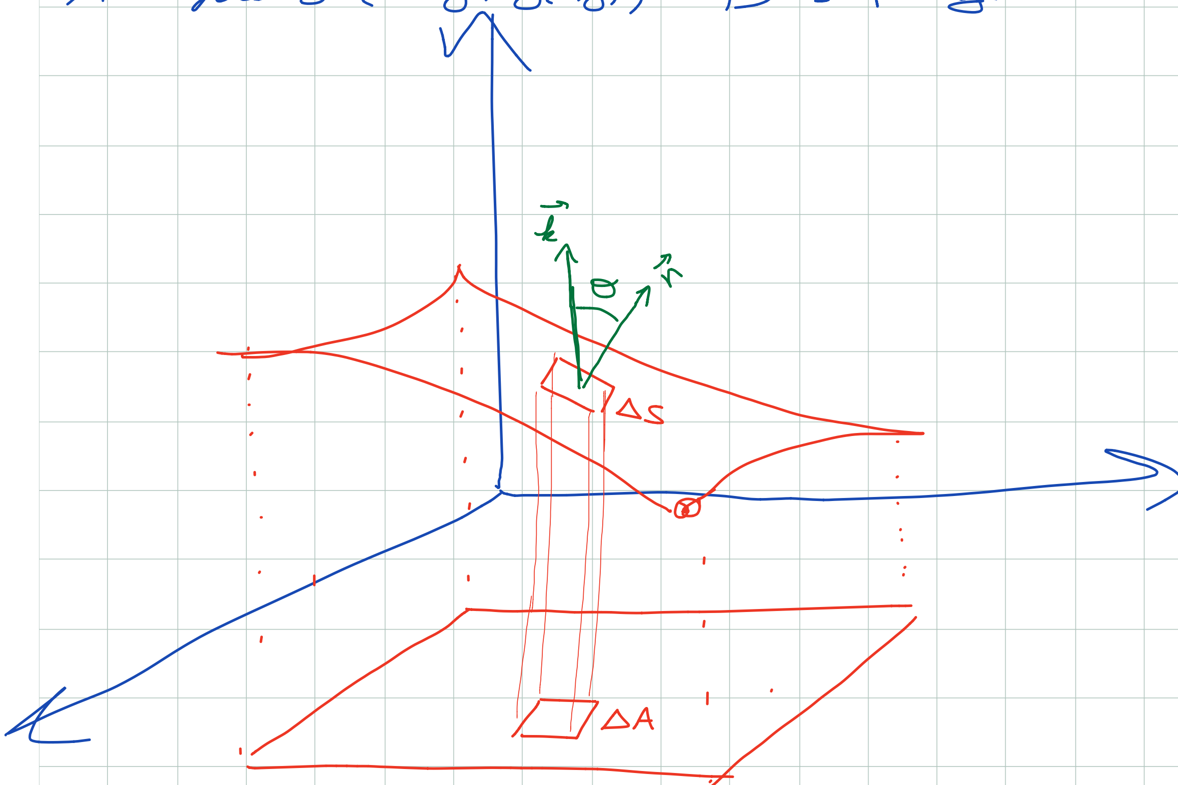
we parametrize: $(x, y, y^2 - x^2)$

$$\iint_S e^{-x}\, dS = \iint_D \frac{x}{\sqrt{4x^2+4y^2+1}} \sqrt{1+(-2x)^2+(2y)^2}\, dx\, dy$$

$$= \int_{-1}^{1}\int_{-1}^{1} x\, dx\, dy = \int_{-1}^{1} x^2/2 \Big|_{-1}^{1} dx\, dy = 0$$

Integrals over graphs (part 2)

As before $S: (x, y, g(x,y))$ D: simple region



here $\vec{n} = \dfrac{\vec{N}}{||\vec{N}||}$ is the unit normal and $\vec{N} = -\dfrac{\partial g}{\partial x}\vec{i} - \dfrac{\partial g}{\partial y}\vec{j} + \vec{k}$ is normal to the surface (exercise: show that $\vec{N}$ is indeed normal to $S: (x, y, g(x,y))$)

Ok, so since $\vec{N}\cdot\vec{k} = \|\vec{N}\|\,\|\vec{k}\|\cos\theta = \|\vec{N}\|\cos\theta$

then $$\cos\theta = \frac{\vec{N}\cdot\vec{k}}{\|\vec{N}\|} = \frac{\left(-\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k}\right)\cdot\vec{k}}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}}$$

$$= \frac{1}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}}$$

But: Recall that
$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\underbrace{\sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}}_{\frac{1}{\cos\theta}}\,dx\,dy$$

So $$\iint_S f(x,y,z)\,dS = \iint \frac{f(x,y,g(x,y))}{\cos\theta}\,dx\,dy$$ ← caution: θ depends on x & y

—x—

<u>Example</u>: Compute the mass of the helicoid $S: (r\cos\theta, r\sin\theta, \theta)$ where $0 \le r \le 1$, $0 \le \theta \le 2\pi$ if its mass density is $m(x,y,z) = \sqrt{x^2+y^2}$

the mass $$M(S) = \iint_S m(x,y,z)\,dS = \iint_D \sqrt{r^2}\,\|T_r \times T_\theta\|\,dr\,d\theta$$

we did this before

$$= \int_0^{2\pi}\int_0^1 r\sqrt{1+r^2}\,dr\,d\theta =$$

$$= \int_0^{2\pi} \frac{2}{3}(r^2+1)^{3/2}\Big|_{r=0}^{r=1} d\theta = \frac{2}{3}\left(2^{3/2}-1\right)\cdot 2\pi$$

Surface integrals of Vector fields

example: The volume of water per unit of time flowing "through a surface" S with velocity given by the field $\vec{F}$ $= \iint_{\Phi} \vec{F} \cdot d\vec{S}$ "$\Phi(D)$"

Definition: The surface integral of $\vec{F}$ over Φ:

$$\iint_{\Phi} \vec{F} \cdot d\vec{S} := \iint_{D} \vec{F} \cdot (\vec{T}_u \times \vec{T}_v)\, du\, dv$$

Interpretation: take the parallelpiped given by $\vec{T}_u \Delta u$ and $\vec{T}_v \Delta v$ and $\vec{F}$. It's volume is $\vec{F} \cdot \underbrace{(\vec{T}_u \times \vec{T}_v) \Delta u \Delta v}_{\Delta \vec{S}}$ and can be seen as the volume of fluid passing through the parallelpiped per unit of time. Summing over all parallelpiped like this and taking $\Delta u, \Delta v \longrightarrow 0$ gives the total volume of fluid flowing through S per unit of time.

$$\text{Flux} = \iint_{\Phi} \vec{F} \cdot d\vec{S} \qquad \text{(terminology)}$$

example: $\vec{F}(x,y,z) = (x, y, z)$

S: sphere of radius 1.

find $\iint_{S} \vec{F} \cdot d\vec{S}$

Sol'n: We parametrize S: $(\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$

$$\Rightarrow T_\theta = (-\sin\theta\sin\phi, \cos\theta\sin\phi, 0)$$
$$T_\phi = (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi)$$

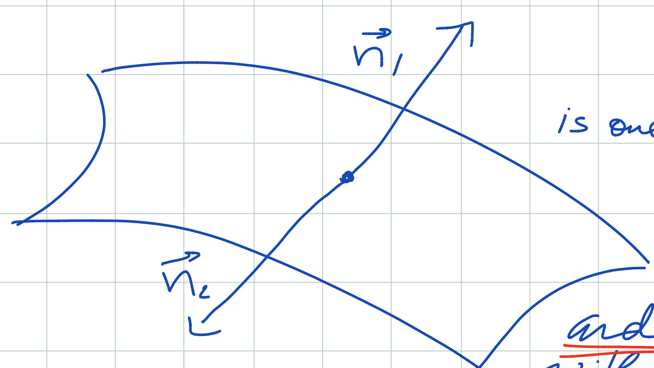
$$\Rightarrow T_\theta \times T_\phi = (-\cos\theta\sin^2\phi, -\sin\theta\sin^2\phi, -\sin\phi\cos\phi)$$

& $\vec{F}$ on the sphere is $(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$

$$\Rightarrow \vec{F}\cdot(\vec{T_\theta}\times\vec{T_\phi}) = -\cos^2\theta\sin^3\phi - \sin^2\theta\sin^3\phi - \sin\phi\cos^2\phi$$
$$= -\sin^3\phi - \sin\phi\cos^2\phi = -\sin\phi$$

$$\Rightarrow \iint_D \vec{F}\cdot(\vec{T_\theta}\times\vec{T_\phi})\, d\theta\, d\phi = \int_0^{2\pi}\int_0^{\pi} -\sin\phi\, d\phi\, d\theta = -4\pi$$

<u>Remark</u>: We implicitly chose an orientation for the surface when we used $\vec{T_\theta}\times\vec{T_\phi}$ instead of $\vec{T_\phi}\times\vec{T_\theta}$. In the fluid flow example this is analogous to asking how much fluid is going "out" of the surface instead of "into" it. <u>So what is orientation?</u>



An oriented surface is one where at each $(x,y,z)\in S$ there are 2 unit normals $\vec{n}_1$ & $\vec{n}_2$ with $\vec{n}_1 = -\vec{n}_2$ <u>and</u> each can be associated with a side of the surface.

Not every surface is orientable: e.g. Mobius strip is not.

Example: The unit sphere can be given an orientation by selecting $\vec{n} = x\vec{i} + y\vec{j} + z\vec{k}$ (this points "outwards")

In the previous example with the sphere our parametrization gave $\vec{T_\theta} \times \vec{T_\phi} = -\vec{n} \underbrace{\sin\phi}_{+ve}$ (why?)

$\Rightarrow$ our parametrization was orientation reversing (vs orientation preserving).

"Theorem": Surface integrals are independant of parametrization (Provided they are orientation preserving).

This is true for scalar fields and vector fields

Theorem: $\iint_S \vec{F} \cdot \underbrace{d\vec{S}}_{(\vec{T_u} \times \vec{T_v})\,du\,dv} = \iint_S \vec{F} \cdot \underbrace{\vec{n}}_{\frac{\vec{T_u} \times \vec{T_v}}{\|\vec{T_u} \times \vec{T_v}\|}} \underbrace{dS}_{\|T_u \times T_v\|\,du\,dv}$

Example: Heat flow: If $T(x,y,z)$ is the temperature at (x,y,z) Then $\nabla T = \frac{\partial T}{\partial x}\vec{i} + \frac{\partial T}{\partial y}\vec{j} + \frac{\partial T}{\partial z}\vec{k}$ is the temperature gradient. $\vec{F} = -k\nabla T$ is a vector field associated with heat flow and $\iint_S \vec{F} \cdot d\vec{S}$ is the flux (or total rate of heat flow) across S.

Suppose $T(x,y,z) = x^2+y^2+z^2$. Find the heat flux across the unit sphere oriented with the outward normal. (use $k=1$)

$\vec{n} = x\vec{i} + y\vec{j} + z\vec{k}$ is the outward normal

$\vec{F} = -\nabla T = -2x\vec{i} - 2y\vec{j} - 2z\vec{k}$

$$\Rightarrow \iint_S \vec{F}\cdot\vec{n}\, dS = -\iint_S (2x\vec{i} + 2y\vec{j} + 2z\vec{k})(x\vec{i} + y\vec{j} + z\vec{k})\, dS$$

$$= -2\iint \underbrace{(x^2+y^2+z^2)}_{1}\, dS = -2\underbrace{A(S)}_{4\pi r^2} = -8\pi$$

Surface integrals over graphs

Suppose that S is the graph of a function, so $S: (x, y, g(x,y))$ and suppose S is oriented with upward pointing normal (i.e. $\vec{k}$ component is +ve)

$$\vec{n} = \frac{-\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k}}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}} \qquad \text{bec.} \quad T_x \times T_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \partial g/\partial x \\ 0 & 1 & \partial g/\partial y \end{vmatrix}$$

Then

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D (F_1\vec{i} + F_2\vec{j} + F_3\vec{k})\cdot\left(-\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k}\right) dx\, dy$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \left(F_1\left(-\frac{\partial g}{\partial x}\right) + F_2\left(-\frac{\partial g}{\partial y}\right) + F_3\right) dx\, dy$$

Example: $z = x^2+y^2$, $x^2+y^2 \le 4$ describes a paraboloid S

Suppose $\vec F = -y\vec i + x\vec j + \vec k$

Compute $\iint_S \vec F \cdot d\vec S$

method (1): S is the graph of function

so $$\iint_S \vec F \cdot d\vec S = \iint_D \left(F_1\left(-\frac{\partial g}{\partial x}\right) + F_2\left(-\frac{\partial g}{\partial y}\right) + F_3\right) dx\,dy$$

$$= \iint_D \left(-y(\cancel{-2x}) + x(\cancel{-2y}) + 1\right) dx\,dy = A(D)$$

$= 4\pi$

method (2): $$\vec T_x \times \vec T_y = \begin{vmatrix} \vec i & \vec j & \vec k \\ 1 & 0 & \partial g/\partial x \\ 0 & 1 & \partial g/\partial y \end{vmatrix} = -\frac{\partial g}{\partial x}\vec i - \frac{\partial g}{\partial y}\vec j + \vec k$$
$$= -2x\vec i - 2y\vec j + \vec k$$

$$\Rightarrow \iint_S \vec F \cdot d\vec S = \dots = 4\pi$$

method (3): $\vec n = \dfrac{\vec T_x \times \vec T_y}{\|T_x \times T_y\|} = \dots$ & $\vec F \cdot \vec n = \dots$ or get $\vec n$ geometrically

$$\Rightarrow \iint_S \vec F \cdot \vec n\, dS = \iint_D \underbrace{(\quad)}_{\vec F \cdot \vec n} \underbrace{\|T_u \times T_v\|\, du\,dv}_{dS}$$